Indian Statistical Institute, Bangalore
B. Math.
First Year, Second Semester
Analysis II

Mid-term Examination Date : Feb. 26, 2018
Maximum marks: 100 Time: 3 hours

Here the set of natural numbers is denoted by $\mathbb{N}$ and the set of real numbers is denoted by $\mathbb{R}$.

1. Compute upper and lower Riemann integrals of following function and determine as to whether it is Riemann integrable or not: Let $f : [0,2] \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} 5 & \text{if } \ x^2 \text{ is rational.} \\ 7 & \text{otherwise.} \end{cases}$$
[15]

2. Let $a < b$ be real numbers. Suppose $f : [a,b] \to \mathbb{R}$ is monotonic. Show that f is Riemann integrable. [15]

3. Let $a < b$ and $c < d$, be real numbers and let $u : [a,b] \to [c,d]$ be a continuously differentiable function. Let $f : [c,d] \to \mathbb{R}$ be a continuous function. Show that
$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(y)dy.$$
[15]

4. Define $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ by $d(m,n) = |\frac{1}{m^2} - \frac{1}{n^2}|$ Show that d defines a metric on $\mathbb{N}$. Show that $\mathbb{N}$ complete with respect to this metric. [15]

5. On $\mathbb{R}^2$, consider the usual metric d, defined by
$$d((x_1,x_2),(y_1,y_2)) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}.$$
For the following subsets A, B of $\mathbb{R}^2$, determine the closure and the interior of the closure.
(i) $A = \{(x_1,x_2) : x_1 + x_2 \text{ is rational. }\}$
(ii) $B = \{(x_1,x_2) : |x_1| < |x_2|\}$.
[15]

6. Denote interior of a subset S of a metric space by S^o. Let C, D be subsets of a metric space (Y,d). Show that $(C \bigcup D)^o \supseteqq C^o \bigcup D^o$. Give an example where $(C \bigcup D)^o \neq C^o \bigcup D^o$. [15]

7. Let (X,d_1) be a metric space. Define d_2 on $X \times X$ by
$$d_2(x,y) = \begin{cases} d_1(x,y) & \text{if } \ 0 \leq d_1(x,y) \leq 1 \\ 1 & \quad \text{Otherwise.} \end{cases}$$
Show that d_2 is a metric on X. Show that a set A is open in (X,d_1) iff it is open in (X,d_2).
[15]